TEMA 1 2º BACH A

1. Halla la matriz X que verifica: $M^2 \cdot X - N = M \cdot X$

Siendo $M = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}$ y $N = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & -1 \end{pmatrix}$

2. Resuelve el siguiente sistema $\left. \begin{array}{l} 3X - 2Y = \begin{pmatrix} 0 & 5 & -4 \\ 5 & 9 & 0 \\ 15 & -4 & 4 \end{pmatrix} \\ 5X + 4Y = \begin{pmatrix} 7 & 1 & 2 \\ -6 & 6 & 7 \\ 10 & -5 & -2 \end{pmatrix} \end{array} \right\}$

3. Una matriz cuadrada se dice que es ortogonal si su inversa coincide con su traspuesta. Calcula a y b para que la matriz A sea ortogonal $A = \begin{pmatrix} 3/5 & a & 0 \\ b & -3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4. Sea la matriz $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -4 & -m \\ 4 & 10 & m^2 \end{pmatrix}$

a) Estudia el rango de B en función de valor de m

b) ¿para qué valores de m la función es regular?

5. Sea la matriz $A = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

a) Encuentra las matrices B que conmuten con A

b) Calcula A^6, A^{52} , A^n

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① $M^2X - N = MX \Rightarrow M^2X - MX = N \Rightarrow (M^2 - M)X = N \Rightarrow X = (M^2 - M)^{-1} \cdot N$

$$M^2 = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 4 & 1 & -2 \\ 2 & 0 & 2 \end{pmatrix}$$

$$M^2 - M = \begin{pmatrix} 2 & -1 & 0 \\ 4 & 1 & -2 \\ 2 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 3 & -1 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 3 & -1 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[3F_1 - F_3]{2F_1 - F_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & 2 & -1 & 0 \\ 0 & -2 & 0 & 3 & 0 & -1 \end{array}\right) \xrightarrow[F_2 - F_3]{-2F_1 + F_2}$$

$$\left(\begin{array}{ccc|ccc} -2 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 4 & 2 & -1 & 0 \\ 0 & 0 & 4 & -1 & -1 & 1 \end{array}\right) \xrightarrow[F_2 - F_3]{2F_1 - F_3} \left(\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & -1 & -1 \\ 0 & -2 & 0 & 3 & 0 & -1 \\ 0 & 0 & 4 & -1 & -1 & 1 \end{array}\right) \longrightarrow$$

$$\xrightarrow[F_3/4]{F_1/-4,\ F_2/-2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & -1/4 & -1/4 & 1/4 \end{array}\right)$$

$$\underbrace{\begin{matrix} -1/4 & 1/4 & 1/4 \\ -3/2 & 0 & 1/2 \\ -1/4 & -1/4 & 1/4 \end{matrix}}_{(M^2 - M)^{-1}}$$

$$X = \begin{pmatrix} -1/4 & 1/4 & 1/4 \\ -3/2 & 0 & 1/2 \\ -1/4 & -1/4 & 1/4 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -1/4 & 5/4 & -2/4 \\ -6/2 & 4/2 & -1/2 \\ -3/4 & -1/4 & 0 \end{pmatrix}$$

② $A = \begin{pmatrix} 0 & 5 & -4 \\ 5 & 9 & 0 \\ 15 & -4 & 4 \end{pmatrix}$ $\quad B = \begin{pmatrix} 7 & 1 & 2 \\ -6 & 6 & 7 \\ 10 & -5 & -2 \end{pmatrix}$

$$\left.\begin{array}{l} 3X - 2Y = A \\ 5X + 4Y = B \end{array}\right\} \quad \begin{array}{l} \cdot(5) \quad \cancel{15X} - 10Y = 5A \\ \cdot(-3) \quad \cancel{-15X} - 12Y = -3B \\ \hline \qquad -22Y = 5A - 3B \end{array} \qquad \begin{array}{l} \cdot(2) \quad 6X - \cancel{4Y} = 2A \\ \qquad 5X + \cancel{4Y} = B \\ \hline \qquad 11X = 2A + B \end{array}$$

$$Y = \frac{5A - 3B}{-22} \qquad X = \frac{2A + B}{11}$$

$$X = \frac{\begin{pmatrix} 0 & 10 & -8 \\ 10 & 18 & 0 \\ 30 & -8 & 8 \end{pmatrix} + \begin{pmatrix} 7 & 1 & 2 \\ -6 & 6 & 7 \\ 10 & -5 & -2 \end{pmatrix}}{11} = \begin{pmatrix} 7/11 & 11/11 & -6/11 \\ 4/11 & 24/11 & 7/11 \\ 40/11 & -13/11 & 6/11 \end{pmatrix}$$

$$Y = \frac{\begin{pmatrix} 0 & 25 & -20 \\ 25 & 45 & 0 \\ 75 & -20 & 20 \end{pmatrix} - \begin{pmatrix} 21 & 3 & 6 \\ -18 & 18 & 21 \\ 30 & -15 & -6 \end{pmatrix}}{-22} = \begin{pmatrix} 21/22 & -22/22 & 26/22 \\ -43/22 & -27/22 & 21/22 \\ -45/22 & 5/22 & -26/22 \end{pmatrix}$$

③ A es ortogonal si $A^{-1} = A^t$

$A \cdot A^{-1} = A \cdot A^t \Longrightarrow I = A \cdot A^t$

$$\begin{pmatrix} 3/5 & a & 0 \\ b & -3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3/5 & b & 0 \\ a & -3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9/25 + a^2 & 3/5 b - 3/5 a & 0 \\ 3/5 b - 3/5 a & b^2 - \frac{9}{25} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 9/25 + a^2 & 3/5 b - 3/5 a & 0 \\ 3/5 b - 3/5 a & b^2 + \frac{9}{25} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} \frac{9}{25} + a^2 = 1 \\ \frac{3}{5}b - \frac{3}{5}a = 0 \\ b^2 + \frac{9}{25} = 1 \end{array}\right\} \to a = b \qquad a^2 = 1 - \frac{9}{25} = \frac{16}{25} \to \boxed{\begin{array}{l} a = \pm \frac{4}{5} \\ a = b \end{array}}$$

Luego las dos soluciones $A_1 = \begin{pmatrix} 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $A_2 = \begin{pmatrix} 3/5 & -4/5 & 0 \\ -4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

④ $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -4 & -m \\ 4 & 10 & m^2 \end{pmatrix} \xrightarrow[4F_1 - F_3]{2F_1 - F_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 6 & 2+m \\ 0 & -6 & 4-m^2 \end{pmatrix} \xrightarrow[F_2+F_3]{} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 6 & 2+m \\ 0 & 0 & -m^2+m+6 \end{pmatrix}$

$$-m^2 + m + 6 = 0 \Rightarrow m = \frac{-1 \pm \sqrt{1+24}}{-2} = \frac{-1 \pm 5}{-2} = \begin{cases} 3 \\ -2 \end{cases}$$

a) Si $m \neq 3,\ m \neq -2 \Rightarrow rg\ B = 3$
$m \in \mathbb{R} - \{3, -2\}$

Si $m = 3,\ m = -2 \Rightarrow rg\ B = 2$

b) B es regular si tiene inversa, es decir, si $rg\ B =$ orden $B = 3$
Luego, será regular si $m \neq 3, m \neq -2$ (Si $m \in \mathbb{R} - \{3, -2\}$).

⑤ a) $AB = BA$ $\quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\left.\begin{array}{l} AB = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 5a+c & 5b+d \end{pmatrix} \\ BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} a+5b & b \\ c+5d & d \end{pmatrix} \end{array}\right\} \Rightarrow \begin{pmatrix} a & b \\ 5a+c & 5b+d \end{pmatrix} = \begin{pmatrix} a+5b & b \\ c+5d & d \end{pmatrix}$$

$$\left.\begin{array}{l} a = a + 5b \\ b = b \\ 5a + c = c + 5d \\ 5b + d = d \end{array}\right\} \begin{array}{l} b = 0 \\ 5a = 5d \Rightarrow a = d \end{array}$$

$B = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} \ \forall a, c \in \mathbb{R}$

b) $A^2 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}$; $A^3 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 15 & 1 \end{pmatrix}$

$A^n = \begin{pmatrix} 1 & 0 \\ 5n & 1 \end{pmatrix}$ Luego $A^6 = \begin{pmatrix} 1 & 0 \\ 30 & 1 \end{pmatrix}$; $A^{52} = \begin{pmatrix} 1 & 0 \\ 260 & 1 \end{pmatrix}$
$\forall n \in \mathbb{R}$.